

Bijections for Weyl Chamber walks ending on an axis, using arc diagrams

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Abstract. There are several examples of enumerative equivalences in the study of lattice walks where a trade-off appears between a stronger domain constraint and a stronger endpoint constraint. We present a strategy, based on arc diagrams, that gives a bijective explanation of this phenomenon for two kinds of 2D walks (simple walks and hesitating walks). For both step sets, on the one hand, the domain is the octant and the endpoint lies on the x -axis and on the other side, the domain is the quadrant and the endpoint is the origin. Our strategy for simple walks extends to any dimension and yields a new bijective connection between standard Young tableaux of height at most $2k$ and certain walks with prescribed endpoints in the k -dimensional Weyl chamber of type D.

1 Introduction

In the context of directed 2D lattice paths with unit steps, there is a classic bijection between *meanders* and *bridges* of equal length. This maps lattice walks with steps $(1, 1)$ and $(1, -1)$ starting at the origin, staying above the x -axis (meanders) to those ending at height zero (bridges) – see Figure 1. This example illustrates a common trade-off in

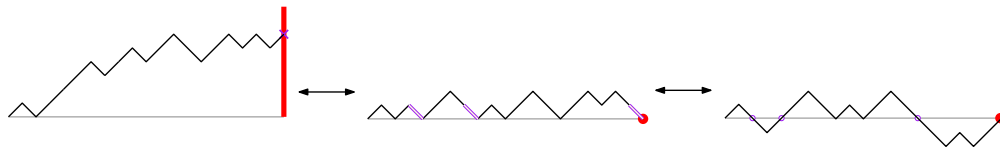


Figure 1: An example of the classical bijection between meanders and bridges.

lattice walks between domain constraints and endpoint constraints [7, 3]. Note that the natural bijection shown in Figure 1 proceeds via an intermediate class of walks where both the stronger domain and endpoint restrictions are imposed, and the elements of this class carry additional “decorations” (here, marked down steps reaching the x -axis).

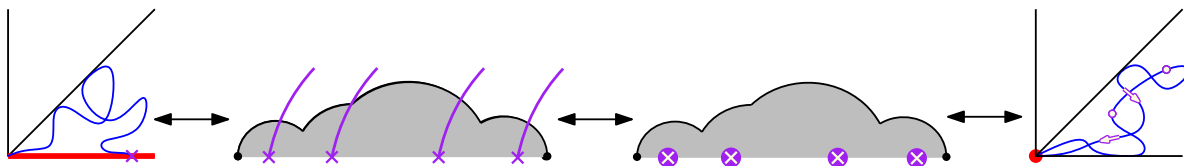


Figure 2: In the first part of our bijections for 2D walks, we map axis-walks in the octant to open arc diagrams with no 3-crossing; mark the positions of the open arcs, and then remove them; then we apply the inverse bijection on the resulting arc diagram with no open arcs, which yields an excursion with some markings.

This work illustrates how a similar strategy can successfully be applied to several models of Weyl chamber walks in arbitrary dimension. In particular, for two classical step sets (simple walks and hesitating walks), we have found an explicit bijection which exchanges a domain constraint with an endpoint constraint. We demonstrate this in the two dimensional case by giving bijections between walks in the quadrant $\{x \geq 0, y \geq 0\}$ ending at the origin (excursions), and walks in the octant $\{x \geq y \geq 0\}$ and ending on the x -axis (axis-walks). For both step sets, the bijections pass through decorated excursions restricted to the octant.

Deciding exactly how to mark the steps in the decorated intermediary is less obvious than the Dyck path example. We do this by using *open arc diagrams* that are associated to the walks via the robust bijection of Chen *et al.* [5], rather, its extension to open arc diagrams due to Burrill *et al.* [4]. In their full generality, these bijections map open diagrams with no $(k + 1)$ -crossing¹ to walks in the k -dimensional Weyl chamber $\{(x_1, \dots, x_k) : x_1 \geq \dots \geq x_k \geq 0\}$ that end on the x_1 -axis, where the number of open arcs gives the abscissa of the endpoint. It is at the level of arc diagrams that the marking of the object is easiest to describe: we map walks that end on the x -axis to open arc diagrams, mark and remove the open arcs, and then apply the inverse bijection to get marked excursions. The schematic outline of our core idea is illustrated in Figure 2. The advantage of our approach is that it very easily generalizes to walks in arbitrary dimension.

In the second part of the bijection, processing the marks yields a walk in a larger domain. This processing is handled differently in the two classes of 2D walks we consider.

1.1 Bijection for 2D simple walks

A lattice model is said to be *simple* if each step is an elementary vector, denoted here by the compass directions $\{N, E, S, W\}$. Our first main result is the following theorem, proved in Section 3.

¹A k -crossing is a set of k mutually crossing arcs.

Theorem 1. *There exists an explicit bijection (preserving the length) between simple axis-walks of even length staying in the first octant, and simple excursions staying in the first quadrant.*

As announced in the introduction, our strategy uses open arc diagrams to make the simple axis-walk of length $2n$ into a decorated excursion. This is then transformed to a simple walk of length $2n$ in the tilted quadrant $\{(x, y) : x \geq 0, |y| \leq x\}$ starting and ending at $(1/2, 1/2)$, and finally mapped to a pair of Dyck paths of respective lengths $2n$ and $2n + 2$. These are known [6, 1] to be in bijection with simple excursions of length $2n$ in the quadrant. This is to compare with the following result recently proved by Elizalde:

Theorem 2 (Elizalde [7]). *There exists an explicit bijection (preserving the length) between simple walks staying in the first octant and ending on the diagonal, and simple excursions staying in the first quadrant.*

In Section 5.1 we provide an alternative proof of Theorem 2 using Schnyder woods. Note that Theorem 1 and 2 together yield a bijection for simple walks of length $2n$ staying in the octant, mapping those ending on the x -axis to those ending on the diagonal. This answers an open question of Bousquet-Mélou and Mishna [3].

Moreover, in Section 5.2 we give an extension for dimension $k \geq 1$ of the aforementioned bijection between simple axis-walks in the octant and simple walks from $(\frac{1}{2}, \frac{1}{2})$ to itself in the tilted quadrant. This yields a new bijective connection between standard Young tableaux of height at most $2k$ and simple walks with prescribed endpoints in the k -dimensional Weyl chamber of type D.

1.2 Bijection for 2D hesitating walks.

A (2-dimensional) *hesitating walk* is a sequence of steps s_1, \dots, s_{2n} such that every step of odd index is either in $\{N, E\}$ (positive step) or is $\mathbf{0} = (0, 0)$, every step of even index is either in $\{W, S\}$ (negative step) or is $\mathbf{0}$, and for every $i \in \{1, \dots, n\}$, s_{2i-1} and s_{2i} cannot both be zero. It is convenient to not represent the null step in the drawings, but rather to group the steps by pairs of the form (s_{2i-1}, s_{2i}) . In Section 4 we show the analogous, although more difficult, result for hesitating walks, which answers a recent question of Burrill *et al.* [4].

Theorem 3. *There exists an explicit bijection (preserving the length) between hesitating axis-walks in the first octant, and hesitating excursions in the first quadrant.*

Again the first step is to use the strategy of Figure 2 to turn the axis-walks into decorated hesitating excursions, where the decoration consists in marking some W-steps on the x -axis. A further ingredient here is to turn the decoration into marked steps leaving the diagonal, after which the decorated excursions in the octant are known [4] to be equivalent to hesitating excursions in the quadrant.

Hesitating excursions of half-length $n - 1$ in the quadrant are known to be counted by the Baxter numbers $B_n = \frac{2}{n(n+1)^2} \sum_{k=1}^n \binom{n+1}{k+1} \binom{n+1}{k} \binom{n+1}{k-1}$. Indeed, as shown in [4], they are in easy bijection with the classical Baxter family of non-intersecting triples of directed lattice walks. On the other hand it has been first shown in [14] (and more recently in [4]) that hesitating axis-walks of half-length n in the octant are also counted by B_{n+1} . Both of these proofs demonstrate an equivalence of generating functions, and neither proof retains significant combinatorial intuition. Our result is the first bijective proof that these walks are counted by B_{n+1} . Such a result is not obvious to find since the family of hesitating axis-walks in the octant does not seem to be naturally endowed with the classical (bivariate) symmetric generating tree

common to the Baxter families such as Baxter permutations, twin pairs of binary trees, 2-oriented plane quadrangulations, and plane bipolar orientations. These families share the same generating tree, and hence there exists a “canonical” bijection relating them. We cannot rely on such a systematic bijective strategy here.

2 Open arc diagrams

Arc diagrams are a graphic representation of combinatorial structures such as partitions or matchings, enabling a convenient visualization of certain patterns, such as crossings. A *partition diagram* is defined for a set partition π of $\{1, \dots, n\}$: draw n points on a line, labeled from 1 to n ; for each (ordered) block $\{a_1, \dots, a_k\}$ of π , we draw an arc from a_i to a_{i+1} for $1 \leq i \leq k - 1$. A *matching diagram* is a partition diagram where the underlying set partition is a matching (i.e. every block has size 2).

A point of a partition diagram can be an *opening point*, if it is the first point of a block of size ≥ 2 ; a *closing point*, if it is the last point of a block of size ≥ 2 ; a *transition point*, if it is a non-extremal point in a block of size ≥ 3 ; or a *fixed point*, if it is a block of size 1. A matching diagram only has opening and closing points.

A *3-crossing pattern* in an arc diagram is a set of 3 mutually crossing arcs, i.e. three arcs (i_1, j_1) , (i_2, j_2) and (i_3, j_3) with $i_1 < i_2 < i_3 < j_1 < j_2 < j_3$. An *enhanced 3-crossing* is a 3-crossing where arcs sharing an endpoint are also considered to be crossing. More formally, three arcs (i_1, j_1) , (i_2, j_2) , (i_3, j_3) form an enhanced 3-crossing if $i_1 < i_2 < i_3 \leq j_1 < j_2 < j_3$. These definitions are naturally generalized to k -crossings for $k \geq 2$.

Arc diagrams can be extended to *open arc diagrams*, by allowing arcs with only a left endpoint, and no right endpoint (see Figures 3 and 4 for examples). In terms of crossings, an open arc is considered to end at an imaginary point to the right, and two open arcs are assumed to be non crossing. For all types of arc diagrams the *size* is defined as the number of points in the diagram.

In [5], Chen *et al.* describe a bijection between arc diagrams with no $k + 1$ -crossings, and excursions staying in the k -dimensional Weyl chamber of type C. It was subse-

quently extended in [4] by Burrill *et al.* to map open arc diagrams to axis-walks.

Theorem 4 (Burrill *et al.* [4] (restricted to 3-crossings)). *There exists an explicit combinatorial bijection between open matching (resp. open partition) diagrams of size n , with m open arcs and no 3-crossing (resp. no enhanced 3-crossing), and simple (resp. hesitating) walks of length n (resp. of half-length n) staying in the first octant $\{(x, y), 0 \leq y \leq x\}$, starting at the origin and ending at $(m, 0)$.*

We refer the reader to [4] for a full description of the bijection. It is based on Robinson-Schensted insertion algorithm on tableaux, but can also conveniently be reformulated in terms of growth diagrams [12].

Here are important properties of this correspondence that we use in our bijections.

Property 5 (Proposition 3 from [4]). *Let π be a closed matching (resp. partition) diagram of size n with no 3-crossing (resp. enhanced 3-crossing), and ω the simple (resp. hesitating) walk of length n corresponding to π via the bijection from [4].*

Open arcs can be inserted into intervals at positions i_1, i_2, \dots, i_k in π without forming a 3-crossing if and only if for every $j \in \{1, \dots, k\}$ the y -coordinate after i_j steps in ω is zero.

For π a partition diagram, the fixed points of π correspond to the factors EW in ω , and the closing points of π correspond to the factors $\{0W, 0S\}$ in ω (with 0 denoting the zero step). In addition, an open arc can be added on a fixed point or a closing point without creating an enhanced 3-crossing if and only if an open arc can be added into the interval before.

3 Proof of Theorem 1: simple walks

The first main ingredient lies in the results from [6, 1], where the respective authors describe a correspondence between simple excursions of length $2n$ in the quadrant, and pairs of Dyck paths of lengths $2n$ and $2n + 2$. To have a bijective proof of Theorem 1, we then need to connect such pairs of Dyck paths to simple axis-walks of even length in the octant. This is given by the following theorem, with an extension to odd length.

Theorem 6. *Let \mathcal{C}_n be the set of Dyck paths of length $2n$, and let \mathcal{U}_n be the set of simple axis-walks of length n in the first octant. There is an explicit bijection for each $n \geq 0$ between \mathcal{U}_{2n} and $\mathcal{C}_n \times \mathcal{C}_{n+1}$, and between \mathcal{U}_{2n+1} and $\mathcal{C}_{n+1} \times \mathcal{C}_{n+1}$.*

This section provides a bijective proof of this result, as illustrated by Figure 3. Gouyou-Beauchamps [10] showed that the cardinality of simple axis-walks in the octant is indeed $\text{Cat}_n \text{Cat}_{n+1}$ or Cat_{n+1}^2 , depending on the parity, where Cat_n is the n th Catalan number. However, his proof, which uses the Gessel-Viennot lemma, involves subtractions and cancellations of terms.

As described in the introduction, our strategy relies on the bijection of Theorem 4, which allows us to turn simple axis-walk into simple excursion with decorations consisting of weights assigned to each visit to the x -axis.

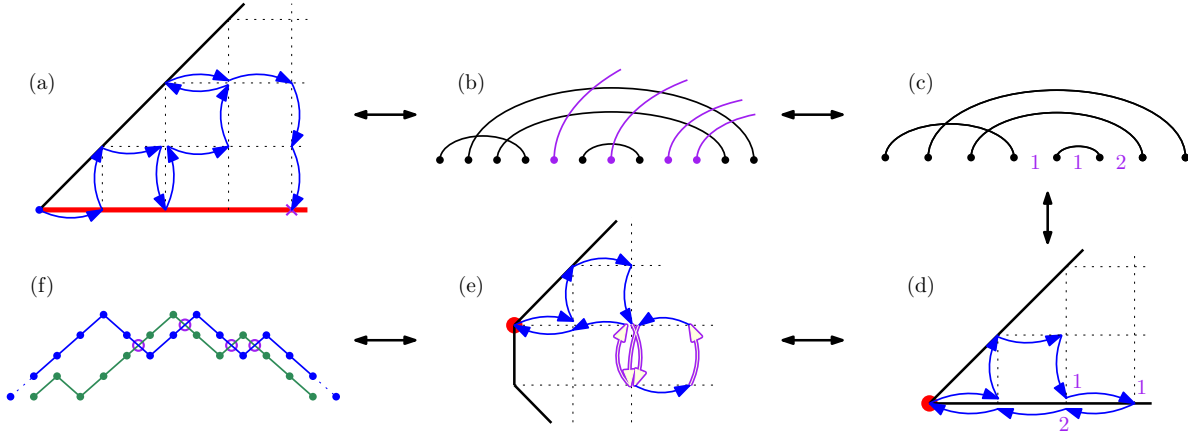


Figure 3: An example of the bijection of Theorem 6. Successive objects are: (a) a simple axis-walk in the octant; (b) an open matching diagram; (c) a matching diagram with no 3-crossing and with integer weights at some intervals to record the positions of the former open arcs; (d) a simple excursion in the octant where each visit to the x -axis carries a nonnegative integer weight; (e) a simple excursion in the tilted quadrant; (f) a pair of Dyck paths whose lengths differ by 2.

Lemma 7. *Simple axis-walks of length n in the octant ending at $(m, 0)$ are in bijection with simple excursions of length $n - m$ in the octant, where each visit to the x -axis carries a non-negative integer weight, such that the weights add up to m .*

Proof. Using Theorem 4, such an axis-walk is mapped to an open matching diagram of size n with m open arcs and without 3-crossing. We then remove the open arcs to obtain a (closed) matching diagram π of size $n - m$, and we record their former positions as follows: for each interval of π that contained at least one open arc, we assign to the interval a positive weight equal to the number of open arcs it formerly contained (see Figure 3(b) to (c)). The sum of these weights is thus m . Note that we cannot insert open arcs in every interval without potentially forming a 3-crossing. This means that only specific intervals can carry weights.

By Theorem 4 (again), the diagram π is mapped to an excursion in the octant. By Property 5, we know that the intervals of π where insertion of open arcs is possible exactly correspond to the visits of the excursion to the x -axis. We then transfer the weights to the corresponding positions (see Figure 3(c) to (d)). \square

As a final step, we transform the weighted excursions in the octant into pairs of Dyck paths. To do so, we define an intermediary class of walks: excursions (or quasi-excursions, depending on the parity of the length) in the tilted quadrant

$$\tilde{Q} = \{(x, y) : x \geq 0, |y| \leq x\},$$

domain which corresponds to the duplication of the octant $\{(x, y) : x \geq y \geq 0\}$ by a symmetry with respect to $y = 0$.

Lemma 8. *For n, m both even (resp. both odd), the set of simple decorated excursions of length $n - m$ in the octant with a total weight m on the visits to the x -axis is in bijection with the set of simple walks of length n in the tilted quadrant \tilde{Q} from $(\frac{1}{2}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{1}{2})$ (resp. to $(\frac{1}{2}, -\frac{1}{2})$) where exactly m steps change the sign of y in the walk. This set is in bijection with $\mathcal{C}_{\lfloor (n+1)/2 \rfloor} \times \mathcal{C}_{\lceil (n+1)/2 \rceil}$, in such a way that if the two Dyck paths are drawn with respective starting points $((-1, 0), (0, 0))$, they cross exactly m times.*

Proof. The idea is illustrated by Figure 3(d)–(f). The weights indicate a switch from one copy of the octant to the other within \tilde{Q} (one copy is for $y > 0$, the other one for $y < 0$). We use the convention that the walk in the lower copy is upside-down.

Concerning the second bijection, we map any walk $(x_i, y_i)_{i \in \{0, \dots, n\}}$ of \tilde{Q} to the pair of paths

$$\left((x_i + y_i)_{i \in \{0, \dots, n\}}, (x_i - y_i)_{i \in \{0, \dots, n\}} \right),$$

which is provably a pair of Dyck paths, once we have adjusted the start and end points to be 0 by adding a positive step or a negative step. \square

Composing Lemma 7 with Lemma 8, we obtain the bijection for Theorem 6.

4 Proof of Theorem 3: hesitating walks

In this abstract we only sketch our bijection between hesitating axis-walks in the octant, and hesitating excursions to prove Theorem 3. The full proof is described in the forthcoming long version of this paper.

4.1 Transformation into decorated hesitating excursions in the octant

The general strategy is the same as for simple walks. We map an axis-walk in the octant to an open partition diagram, remove open arcs noting their location, and then convert to decorated excursions. The second part is different from the simple walk case, as we need an additional step of decoration transfer.

Lemma 9. *Hesitating walks of length $2n$ staying in the octant and ending at $(m, 0)$ are in bijection with hesitating excursions of length $2n$ staying in the octant in which m W -steps on the x -axis have been marked.*

Proof. Using Property 5 it is easily verified that, for π a partition-diagram of size n , and ω the corresponding hesitating excursion of length $2n$ in the octant, the points (closing or fixed) of π where an open arc can be added exactly correspond to the W -steps of ω on the x -axis. If we mark m such steps we obtain an open partition diagram of size n with m open arcs and no enhanced 3-crossing, which itself corresponds (by Theorem 4) to an hesitating walk of length $2n$ in the octant that ends at $(m, 0)$. \square

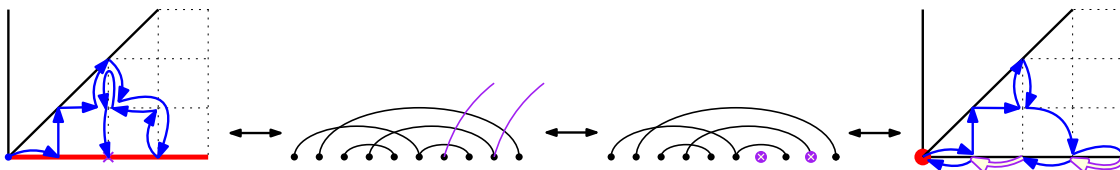


Figure 4: An example of the first part of the bijection. From left to right: a hesitating axis-walk in the octant; an open partition diagram; a decorated partition diagram; a decorated hesitating excursion in the octant

A similar property can be deduced for hesitating excursions in the first quadrant. The following results is proved in [4] and can be seen as a consequence of the reflection principle with respect to the diagonal.

Lemma 10. *Hesitating (resp. simple) excursions of length $2n$ in the first quadrant are in bijection with hesitating (resp. simple) excursions of length $2n$ in the first octant with marked steps leaving the diagonal $y = x$.*

The number of marked steps of the second object corresponds either to a parameter called switch-multiplicity, which is roughly speaking the number of times that the walk crosses the diagonal, or similarly to the number of times the walk goes over the diagonal.

4.2 Moving the marks around

In view of the two previous lemmas, we can see that Theorem 3 holds as soon as the parameters counting the steps leaving the diagonal on one hand, and W -steps on the x -axis on the other hand, are equidistributed. It is in fact the case, and they are even symmetric.

Proposition 11. *There is an explicit involution over the set of hesitating excursions of length $2n$ in the octant that exchanges the number of W -steps on the x -axis and the number of steps leaving the diagonal.*

Proof (sketch).

From hesitating walks to simple walks. We transform every hesitating walk into a simple walk in which some *sailing points* (that is, when a positive step, E or N, is followed by a negative step, W or S) are marked. To do so, we gather the steps of the hesitating walk in pairs, discarding the zero-steps, and marking every sailing point induced by the gathering of two non-zero steps. Thus, every hesitating excursion in the octant identifies to a simple excursion in the octant where some sailing points are marked.

From simple walks to pairs of Dyck paths. Simple excursions in the octant are mapped to non-crossing pairs of Dyck paths thanks to the transformation $((x_i + y_i), (x_i - y_i))$.

The sailing points of the excursion become the peaks of the upper path; the W-steps on the x -axis become the *upper bounces*, i.e. down steps occurring at the same time and the same height for both paths; and the steps leaving the diagonal become the *lower bounces*, i.e. up steps leaving the x -axis in the lower path. We are thus reduced to find an involution on non-crossing pairs of Dyck paths that preserves the number of upper peaks and exchanges the number of upper bounces with the number of lower bounces.

Involution for non-crossing pairs of Dyck paths. The involution is given by an article of Elizalde and Rubey [8]. More specifically, this involution operates on non-crossing pairs of Dyck paths, preserves the upper path (hence the upper peaks), and exchanges the number of upper and lower bounces. We found an alternative proof using Schnyder woods (see next section).

5 Further results

5.1 Alternative proofs using Schnyder woods

A *planar triangulation* is a simple planar graph embedded in the plane where all faces are triangular. The three vertices in counterclockwise order around the external face are denoted v_0, v_1, v_2 . A *Schnyder wood* is a specific partition (see [2] for a detailed definition and references) of the internal edges of a planar triangulation into a blue tree T_0 , a red tree T_1 , and a green tree T_2 such that for $i \in \{0, 1, 2\}$, T_i spans all internal vertices and is rooted at v_i .

Theorem 12 (Bernardi, Bonichon [2]). *Non-crossing pairs of Dyck paths of length $2n$ are in bijection with Schnyder woods of size n .*

In the course of this bijection, $\deg(v_0) - 2$ is mapped to the number of lower bounces, and $\deg(v_1) - 2$ is mapped to the length of the final descent of the upper Dyck path. We can then exchange the roles of T_0 and T_1 and obtain an involution on non-crossing pairs of Dyck paths that exchanges the number of lower bounces with the length of the last descent of the upper Dyck path. Hence, if we denote by $\mathcal{N}_{n,i,j}$ the set of non-crossing pairs of Dyck paths of length $2n$, with i lower bounces, and where the last descent of the upper Dyck path has length j , then we have $\sum_i \mathcal{N}_{n,i,j} 2^i \simeq \sum_j \mathcal{N}_{n,i,j} 2^j$. As it turns out (see Figure 5 for an example), the first set is in bijection with simple excursions of length $2n$ staying in the first quadrant, while the second set is in bijection with simple walks of length $2n$ staying in the first octant and ending on the diagonal. We thus obtain a new proof of Theorem 2.

We can also easily extract an involution on non-crossing pairs of Dyck paths that preserves the number of upper peaks (peaks in the upper path), while exchanging the number of upper bounces and the number of lower bounces (required to conclude the

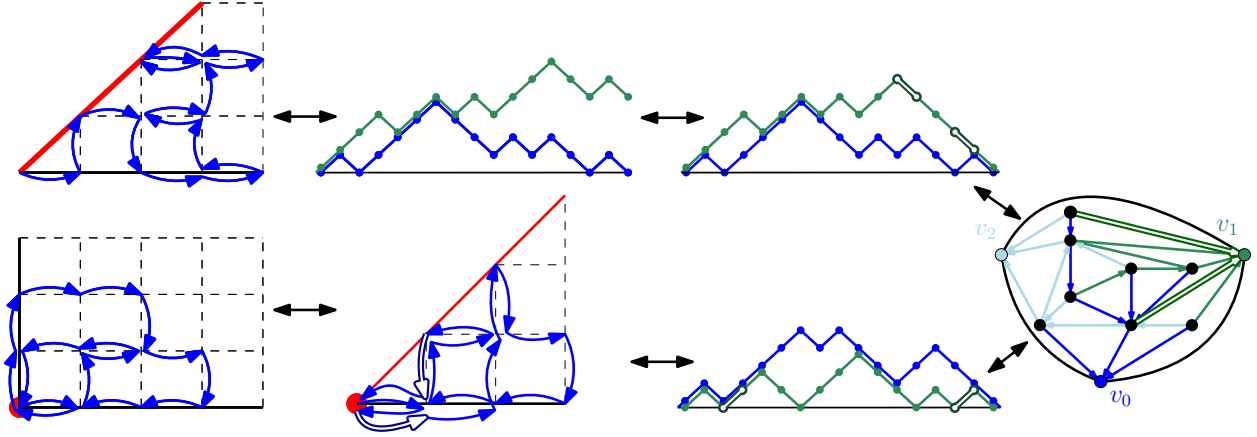


Figure 5: Illustration of the bijective proof of Theorem 2 using Schnyder woods. The top-part shows how a simple walk ending on the diagonal and staying in the octant identifies to a non-crossing pair of Dyck paths where some steps on the last descent of the upper path are marked. The lower-part shows how a simple excursion staying in the quadrant identifies to a non-crossing pair of Dyck paths where some up-steps of the lower path that leave the x -axis are marked. Both kinds of walks are in bijection via Schnyder woods, by exchanging the roles of T_0 and T_1 .

proof of Proposition 11). Indeed it can be checked that $\deg(v_2) - 2$ corresponds to the number of upper bounces, and the number of internal nodes of T_1 corresponds to the number of upper peaks. Hence, by exchanging the roles of T_0 and T_2 we obtain the desired involution.

5.2 A new bijection for Young tableaux of even-bounded height

As we have seen in Section 3, the main step to prove Theorem 1 is an explicit bijection between simple axis-walks of length n staying in the octant $\{x \geq y \geq 0\}$, and simple walks of length n from $(\frac{1}{2}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{(-1)^n}{2})$ staying in the tilted quadrant. As it turns out, this bijection can be easily generalized to any dimension, and infers new connections with standard Young tableaux with even-bounded height. Precisely, for $k \geq 1$, the k -dimensional Weyl chamber² of type C is $W_C(k) := \{(x_1, x_2, \dots, x_k) \mid x_1 \geq x_2 \geq \dots \geq x_k \geq 0\}$, and the k -dimensional Weyl chamber of type D is

$$W_D(k) := \{(x_1, x_2, \dots, x_k) \mid x_1 \geq x_2 \geq \dots \geq x_{k-1} \geq |x_k|\}$$

(note that $W_C(2)$ is the first octant and $W_D(2)$ is the tilted quadrant). A walk starting at the origin and ending on the x_1 -axis is called an *axis-walk*. As an extension to any k of the above-mentioned result we obtain the following theorem.

²For convenience we define the chambers using non-strict inequalities, our bijective statements can equivalently be given under strict inequalities, upon applying the coordinate-shift $\tilde{x}_i = x_i + k + 1 - i$.

Theorem 13. *For $k \geq 1$ and $n \geq 0$, there is an explicit bijection between simple axis-walks of length n staying in $W_C(k)$ and simple walks of length n staying in $W_D(k)$, starting from $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})$, and ending at $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{(-1)^n}{2})$. The ending x_1 -coordinate corresponds to the number of steps that change the sign of x_k .*

The arguments are very similar to those in the proofs of Lemma 7 and the first part of Lemma 8. They use the general formulation of Theorem 4 (bijection between open matching diagrams without $(k+1)$ -crossing and simple axis-walks in $W_C(k)$), and the property that the intervals where an open arc can be added (without creating a $(k+1)$ -crossing) correspond to the visits of the walk to $\{x_k = 0\}$.

For $n, d \geq 1$, let $\mathcal{Y}_n^{(d)}$ be the set of standard Young tableaux of size n with height at most d . It has been recently shown [4, 13] that $\mathcal{Y}_n^{(2k)}$ is in bijection with simple axis-walks of length n in W_C , with the ending x_1 -coordinate mapped to the number of columns of odd length. Composing this bijection with Theorem 13 we obtain:

Corollary 14. *For $n, k \geq 1$, there is an explicit bijection between $\mathcal{Y}_n^{(2k)}$ and simple axis-walks of length n staying in $W_D(k)$, starting from $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})$, and ending at $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{(-1)^n}{2})$. The number of odd columns corresponds to the number of steps that change the sign of x_k .*

Thanks to the lattice path enumeration techniques of Grabiner and Magyar [11]), the previous corollary has an interesting consequence: a combinatorial interpretation of the determinant expression of Gessel [9] for the generating function of standard Young tableaux of even-bounded height

$$\sum_{n \geq 0} \frac{1}{n!} |\mathcal{Y}_n^{(2k)}| x^n = \det (I_{i-j}(2x) + I_{i+j-1}(2x))_{1 \leq i, j \leq k}, \quad (5.1)$$

where $I_m(2x)$ is defined as $\sum_{n \geq 0} \frac{x^{2n+|m|}}{n!(n+|m|)!}$ (for $m \in \mathbb{Z}$).

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